

The space of hypersurfaces singular along a specified curve

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Abstract

Let k be an algebraically closed field, and let $C \subset \mathbb{P}_k^n$ be a fixed reduced closed subscheme. We study the linear space $(W_C)_l$ consisting of hypersurfaces $F \in k[x_0, \dots, x_n]_l$ of degree l which are singular along C . It turns out that C gives rise to a certain ideal sheaf \mathcal{J} such that $(W_C)_l = \Gamma(\mathbb{P}^n, \mathcal{J}(l))$. When C is a curve, we compute the Hilbert polynomial of \mathcal{J} in terms of invariants of C . As a consequence, we show that for any integral plane curve $C \hookrightarrow \mathbb{P}^2 \subset \mathbb{P}^n$, the Hilbert polynomial of the sheaf Ω_C of Kahler differentials is $\chi(\Omega_C(l)) = dl + p_a - 1$, where d and p_a are the degree and arithmetic genus of C , respectively.

1 Introduction

Let k be an algebraically closed field. Let $C \subset \mathbb{P}_k^n$ be a fixed reduced closed subscheme. If $F \in k[x_0, \dots, x_n]_l$ is a hypersurface of degree l , the question that we address in this paper (Section 2) is the following: when is the hypersurface $V(F)$ singular along the subscheme C ? The naive guess is that this happens precisely when $F \in \Gamma(\mathbb{P}^n, \mathcal{I}^2)$, where \mathcal{I} is the ideal sheaf of C . This is correct when C is a local complete intersection. In general, we attach a certain ideal sheaf $\mathcal{J} \supset \mathcal{I}^2$ with the property that for $F \in k[x_0, \dots, x_n]_l$, we have $C \subset V(F)_{\text{sing}}$ if and only if $F \in \Gamma(\mathbb{P}^n, \mathcal{J}(l))$. Next, in the case when C is a curve, the question of computing the Hilbert polynomial of \mathcal{J} appears naturally, and we address it in Section 3.

Let $S = k[x_0, \dots, x_n]$ with the usual grading. Consider the functor

$$\begin{aligned} \widetilde{\Gamma}: \text{QCoh}(\mathbb{P}^n) &\rightarrow \text{Graded } S\text{-modules}, \\ \mathcal{F} &\mapsto \bigoplus_{l \in \mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{F}(l)). \end{aligned}$$

We denote its left adjoint by $\widetilde{\text{Loc}}$ (denoted \sim on p. 116 in [2]). Recall that $\widetilde{\text{Loc}} \widetilde{\Gamma} \xrightarrow{\sim} \text{id}$ and that if M is a finitely-generated graded S -module, then $M \rightarrow \widetilde{\Gamma} \widetilde{\text{Loc}}(M)$ is an isomorphism in large degrees.

2 Understanding the condition $C \subset V(F)_{\text{sing}}$ for a fixed C

Let $I \subset A = k[x_1, \dots, x_n]$ be a radical ideal, and $C = \text{Spec}(A/I)$. For $f \in A, f \neq 0$, we have $C \subset V(f)_{\text{sing}}$ if and only if $f \in \mathfrak{m}^2$ for any maximal $\mathfrak{m} \supset I$ (here, $V(f)_{\text{sing}}$ is the set of singular points of $V(f) = \text{Spec}(A/f)$). We now have to understand this condition.

Lemma 2.1. *For $f \in I$, we have $f \in \mathfrak{m}^2$ for all $\mathfrak{m} \supset I$ if and only if f belongs to the kernel of*

$$I \rightarrow \Omega_{A/k}/I\Omega_{A/k}.$$

Proof. Suppose that f satisfies $f \in \mathfrak{m}^2$ for all $\mathfrak{m} \supset I$. Let $B = A/I$. We claim that f belongs to the kernel of the map $I \rightarrow \Omega_{A/k}/I\Omega_{A/k}$; i.e., we claim

$$df \in I\Omega_{A/k},$$

where $d: A \rightarrow \Omega_{A/k}$ is the canonical derivation. (In this way, we linearize our unhandy condition that $f \in \mathfrak{m}^2$). We know that for each maximal $\mathfrak{m} \supset I$, we have $\Omega_{A/k}/\mathfrak{m}\Omega_{A/k} = \Omega_{A/k} \otimes_A A/\mathfrak{m} \simeq \mathfrak{m}/\mathfrak{m}^2$ as A/\mathfrak{m} -vector spaces, and

$$\begin{array}{ccc} I/I^2 & \longrightarrow & \mathfrak{m}/\mathfrak{m}^2 \\ d \downarrow & & \simeq \downarrow \\ \Omega_{A/k}/I\Omega_{A/k} & \longrightarrow & \Omega_{A/k}/\mathfrak{m}\Omega_{A/k} \end{array}$$

commutes, so the condition $f \in \mathfrak{m}^2$ is equivalent to $df \in \mathfrak{m}\Omega_{A/k}$.

Since $\Omega_{A/k}$ is a free A -module, we conclude that

$$df \in \bigcap_{\mathfrak{m} \supset I} (\mathfrak{m}\Omega_{A/k}) = \left(\bigcap_{\mathfrak{m} \supset I} \mathfrak{m} \right) \Omega_{A/k} = I\Omega_{A/k}.$$

The converse is obvious from the commutative diagram above. \square

Let $i: C \hookrightarrow \mathbb{P}^n$ be any reduced closed subscheme (not necessarily integral or 1-dimensional), and let \mathcal{I} be its ideal sheaf. Define $\mathcal{G} \in \text{Coh}(C)$ as the kernel of the first map in the second fundamental exact sequence,

$$0 \rightarrow \mathcal{G} \rightarrow i^*\mathcal{I} \rightarrow i^*\Omega_{\mathbb{P}^n} \rightarrow \Omega_C \rightarrow 0,$$

and $\mathcal{H} \in \text{Coh}(C)$ by the exactness of

$$0 \rightarrow \mathcal{H} \rightarrow i^*\Omega_{\mathbb{P}^n} \rightarrow \Omega_C \rightarrow 0,$$

so we have a short exact sequence

$$0 \rightarrow \mathcal{G} \rightarrow i^*\mathcal{I} \rightarrow \mathcal{H} \rightarrow 0.$$

Since i_* is exact (as i is a closed embedding, hence affine), it follows that $i_*i^*\mathcal{I} \twoheadrightarrow i_*\mathcal{H}$ is surjective, and hence so is the composition $\mathcal{I} \twoheadrightarrow \mathcal{I}/\mathcal{I}^2 = i_*i^*\mathcal{I} \twoheadrightarrow i_*\mathcal{H}$. Let \mathcal{J} denote its kernel, so we have a short exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{I} \rightarrow i_*\mathcal{H} \rightarrow 0.$$

In other words, \mathcal{J} is defined by the exactness of

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{I} \rightarrow \Omega_{\mathbb{P}^n}/I\Omega_{\mathbb{P}^n} \rightarrow i_*\Omega_C \rightarrow 0.$$

Note that $\mathcal{I}^2 \subset \mathcal{J}$, and we have an equality if C is a local complete intersection, since in this case, $\mathcal{G} = 0$ (see Exercise 16.17 in [1]).

Proposition 2.2. *With notation as above, for $F \in S_{\text{homog}}$, we have $C \subset V(F)_{\text{sing}}$ if and only if $F \in \tilde{\Gamma}(\mathcal{J})$.*

Proof. For $i = 0, \dots, n$, let $f_i = F(x_0, \dots, 1, \dots, x_n)$ be the i -th dehomogenization of F . Assume that $F \neq 0$, so also $f_i \neq 0$. The condition $F \in \tilde{\Gamma}(\mathcal{J})$ is equivalent to $f_i \in \Gamma(D_+(x_i), \mathcal{J})$ for all $i = 0, \dots, n$. On the other hand, $C \subset V(F)_{\text{sing}}$ is equivalent to $C \cap D_+(x_i) \subset V(f_i)_{\text{sing}}$, and hence the statement of the proposition reduces to the following affine statement.

Let $A = k[x_1, \dots, x_n]$ and $C = \text{Spec}(A/I)$, where $I \subset A$ is radical. Let $B = A/I$. Define an A -module J by the exactness of

$$0 \rightarrow J \rightarrow I \rightarrow \Omega_A/I\Omega_A \rightarrow \Omega_B \rightarrow 0.$$

Then for a nonzero polynomial $f \in A$, we have $C \subset V(f)_{\text{sing}}$ if and only if $f \in J$. This follows from Lemma 2.1. \square

Corollary 2.3. *Suppose that $C \subset \mathbb{P}^n$ is a reduced closed subscheme, which is a local complete intersection. Let \mathcal{I} be the ideal sheaf of C . Then for $F \in k[x_0, \dots, x_n]_l$ we have $C \subset V(F)_{\text{sing}}$ if and only if $F \in \Gamma(\mathbb{P}^n, \mathcal{I}^2(l))$.*

Proof. We already observed that when C is a local complete intersection, $\mathcal{J} = \mathcal{I}^2$. \square

3 Computing the Hilbert polynomial of \mathcal{J}

Lemma 3.1. *Let*

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

be a short exact sequence in $\text{QCoh}(\mathbb{P}^n)$, with \mathcal{F}' coherent. Then for large l ,

$$0 \rightarrow \tilde{\Gamma}(\mathcal{F}')_l \rightarrow \tilde{\Gamma}(\mathcal{F})_l \rightarrow \tilde{\Gamma}(\mathcal{F}'')_l \rightarrow 0$$

is a short exact sequence of k -vector spaces.

Proof. Apply Theorem III.5.2(b) in [2] to \mathcal{F}' . \square

Let C be any integral curve over k (not necessarily projective for now), and let $p: \tilde{C} \rightarrow C$ be its normalization. Consider the canonical map $\alpha: \Omega_C \rightarrow p_*\Omega_{\tilde{C}}$. Let $\mathcal{R}_1, \mathcal{R}_2 \in \text{Coh}(C)$ denote its kernel and cokernel:

$$0 \rightarrow \mathcal{R}_1 \rightarrow \Omega_C \xrightarrow{\alpha} p_*\Omega_{\tilde{C}} \rightarrow \mathcal{R}_2 \rightarrow 0.$$

Since p is an isomorphism over a dense open $U \subset C$, so is α , and hence \mathcal{R}_1 and \mathcal{R}_2 have finite support, contained in C_{sing} . For each $P \in C_{\text{sing}}$, the stalks $(\mathcal{R}_1)_P$ and $(\mathcal{R}_2)_P$ are finite-dimensional k -vector spaces.

Define

$$\mu(C) := \sum_{P \in C_{\text{sing}}} (\dim_k(\mathcal{R}_1)_P - \dim_k(\mathcal{R}_2)_P).$$

For the rest of this chapter, let $i: C \hookrightarrow \mathbb{P}^n$ be an integral curve of degree d , and let \tilde{g} be the genus of its normalization.

Lemma 3.2. *For large l ,*

$$\dim_k \Gamma(C, \Omega_C(l)) = dl + \tilde{g} - 1 + \mu(C).$$

Proof. Consider the exact sequence

$$0 \rightarrow \mathcal{R}_1 \rightarrow \Omega_C \xrightarrow{\alpha} p_*\Omega_{\tilde{C}} \rightarrow \mathcal{R}_2 \rightarrow 0.$$

For large l , the sequence

$$0 \rightarrow \Gamma(C, \mathcal{R}_1(l)) \rightarrow \Gamma(C, \Omega_C(l)) \rightarrow \Gamma(C, (p_*\Omega_{\tilde{C}})(l)) \rightarrow \Gamma(C, \mathcal{R}_2(l)) \rightarrow 0 \quad (1)$$

is exact.

Note that

$$\Gamma(C, \mathcal{R}_1(l)) \simeq \Gamma(C, \mathcal{R}_1) = \bigoplus_{P \in C_{\text{sing}}} (\mathcal{R}_1)_P,$$

and similarly for \mathcal{R}_2 .

Now, we look at the term $\Gamma(C, (p_*\Omega_{\tilde{C}})(l))$. By the projection formula, we know

$$(p_*\Omega_{\tilde{C}})(l) \simeq p_*(\Omega_{\tilde{C}} \otimes_{\mathcal{O}_{\tilde{C}}} p^*\mathcal{O}_C(l)).$$

Since C has degree d , $p^*\mathcal{O}_C(l)$ is a line bundle on \tilde{C} of degree dl (see Corollary 5.8 on p. 306 in [3]). By the Riemann-Roch theorem applied to \tilde{C} , it follows that for large l ,

$$\dim_k \Gamma(\tilde{C}, \Omega_{\tilde{C}} \otimes p^*\mathcal{O}_C(l)) = dl + \tilde{g} - 1.$$

Take the alternating sum of dimensions in (1). □

For an integral curve $i: C \hookrightarrow \mathbb{P}^n$ with ideal sheaf \mathcal{I} and $I = \tilde{\Gamma}(\mathcal{I})$, we let d be its degree and p_a be its arithmetic genus, so for large l , we have

$$\dim_k(S/I)_l = dl + 1 - p_a.$$

For $l \geq 1$, let

$$(W_C)_l = \{F \in S_l \mid C \subset V(F)_{\text{sing}}\}.$$

Proposition 3.3. *Let C, i, \tilde{g} be as in Lemma 3.2. For $l \gg 0$,*

$$\dim_k(S_l/(W_C)_l) = ndl + 1 + (n+1)(1-d-p_a) - \tilde{g} - \mu(C).$$

Proof. Recall the short exact sequences

$$0 \rightarrow \mathcal{H} \rightarrow i^*\Omega_{\mathbb{P}^n} \rightarrow \Omega_C \rightarrow 0 \quad (\text{definition of } \mathcal{H}) \quad (2)$$

and

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{I} \rightarrow i_*\mathcal{H} \rightarrow 0 \quad (\text{definition of } \mathcal{J}).$$

We will be using Lemma 3.1 continuously without explicit notice. By Proposition 2.2, for all l , we have $(W_C)_l = \tilde{\Gamma}(\mathcal{J})_l$, so we have to compute $\dim_k \tilde{\Gamma}(\mathcal{O}_{\mathbb{P}^n}/\mathcal{J})_l$ for large l . From the short exact sequence

$$0 \rightarrow i_*\mathcal{H} \rightarrow \mathcal{O}_{\mathbb{P}^n}/\mathcal{J} \rightarrow \mathcal{O}_{\mathbb{P}^n}/\mathcal{I} \rightarrow 0,$$

we obtain a short exact sequence

$$0 \rightarrow \tilde{\Gamma}(i_*\mathcal{H})_l \rightarrow \tilde{\Gamma}(\mathcal{O}_{\mathbb{P}^n}/\mathcal{J})_l \rightarrow \tilde{\Gamma}(\mathcal{O}_{\mathbb{P}^n}/\mathcal{I})_l \rightarrow 0$$

for large l . Since the last term is $(\tilde{\Gamma}(\mathcal{O}_{\mathbb{P}^n})/\tilde{\Gamma}(\mathcal{I}))_l = (S/I)_l$ for large l , and hence of dimension $dl + 1 - p_a$, it suffices to compute the dimension of the first term.

Applying the exact functor i_* to (2), we obtain short exact

$$0 \rightarrow i_*\mathcal{H} \rightarrow \Omega_{\mathbb{P}^n}/\mathcal{I}\Omega_{\mathbb{P}^n} \rightarrow i_*\Omega_C \rightarrow 0,$$

which for large l gives short exact

$$0 \rightarrow \tilde{\Gamma}(i_*\mathcal{H})_l \rightarrow \tilde{\Gamma}(\Omega_{\mathbb{P}^n}/\mathcal{I}\Omega_{\mathbb{P}^n})_l \rightarrow \Gamma(C, \Omega_C(l)) \rightarrow 0.$$

We know the last term has dimension $dl + \tilde{g} - 1 + \mu(C)$.

Finally, we have to compute $\dim_k \tilde{\Gamma}(\Omega_{\mathbb{P}^n}/\mathcal{I}\Omega_{\mathbb{P}^n})_l$ for large l . Recall from Theorem II.8.13 in [2] the short exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow 0.$$

Since $\mathcal{O}_{\mathbb{P}^n}$ is locally free, applying $-\otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}$ to this short exact sequence yields a short exact sequence

$$0 \rightarrow \frac{\Omega_{\mathbb{P}^n}}{\mathcal{I}\Omega_{\mathbb{P}^n}} \rightarrow \left(\frac{\mathcal{O}_{\mathbb{P}^n}(-1)}{\mathcal{I}\mathcal{O}_{\mathbb{P}^n}(-1)} \right)^{\oplus(n+1)} \rightarrow \frac{\mathcal{O}_{\mathbb{P}^n}}{\mathcal{I}} \rightarrow 0.$$

Once again by Lemma 3.1, we know that

$$0 \rightarrow \tilde{\Gamma} \left(\frac{\Omega_{\mathbb{P}^n}}{\mathcal{I}\Omega_{\mathbb{P}^n}} \right)_l \rightarrow \tilde{\Gamma} \left(\frac{\mathcal{O}_{\mathbb{P}^n}(-1)}{\mathcal{I}\mathcal{O}_{\mathbb{P}^n}(-1)} \right)_l^{\oplus(n+1)} \rightarrow \tilde{\Gamma} \left(\frac{\mathcal{O}_{\mathbb{P}^n}}{\mathcal{I}} \right)_l \rightarrow 0$$

is exact for large l . Again, we have to compute the dimensions of the second and third terms. For large l , the third term has dimension $dl + 1 - p_a$, as before.

We are left to compute $\dim_k \tilde{\Gamma}(\mathcal{O}(-1)/\mathcal{I}\mathcal{O}(-1))_l$ for large l . Notice that $\mathcal{I}\mathcal{O}(-1) \simeq \mathcal{I}(-1)$ and that for large l , $\tilde{\Gamma}(\mathcal{O}(-1)/\mathcal{I}(-1))_l$ is $\left(\tilde{\Gamma}(\mathcal{O}(-1))/\tilde{\Gamma}(\mathcal{I}(-1)) \right)_l$, which is $(S/I)_{l-1}$ for large l . This is a shift of S/I and thus dimension equal to $d(l-1) + 1 - p_a$.

Going back through the exact sequences, we complete the calculation. \square

Let $I = (f, x_{b+2}, \dots, x_n) \subset S = k[x_0, \dots, x_n]$, where $f \in k[x_0, \dots, x_{b+1}]_d - \{0\}$. Consider the composition

$$\Phi: k[x_0, \dots, x_{b+1}]_l \oplus \left(\bigoplus_{i=b+2}^n k[x_0, \dots, x_{b+1}]_{l-1} x_i \right) \hookrightarrow S_l \twoheadrightarrow S_l / (I^2 \cap S_l).$$

Note that Φ is surjective.

Lemma 3.4. *We have that*

$$\ker(\Phi) = \{P + \sum_{i=b+2}^n P_i x_i : f^2 | P, f | P_i \text{ for } i = b+2, \dots, n\}.$$

For $l \geq 2d$, the codimension of I_l^2 in S_l equals $\beta_d(l)$, where

$$\beta_d(l) = \binom{l+b+1}{b+1} - \binom{l-2d+b+1}{b+1} + (n-b-1) \left(\binom{l+b}{b+1} - \binom{l-d+b}{b+1} \right)$$

Proof. If $P + \sum P_i x_i \in \ker(\Phi)$, then we can write $P + \sum P_i x_i = T \in I^2$. Expand both sides as polynomials in x_{b+2}, \dots, x_n and just compare the two expressions. The second part is an immediate consequence. \square

Corollary 3.5. *For an integral plane curve C , we have $\mu(C) = p_a - \tilde{g}$.*

However, this fails for a general integral curve C .

Proof. Compare Proposition 3.3 with Lemma 3.4 (combined with Proposition 2.2). \square

Lemma 3.2 and the previous corollary lead to the following result:

Theorem 3.6. *For any integral plane curve $C \hookrightarrow \mathbb{P}^2 \subset \mathbb{P}^n$, the Hilbert polynomial of the sheaf Ω_C of Kähler differentials is*

$$\chi(\Omega_C(l)) = dl + p_a - 1.$$

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References

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